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# Boundary conjugation problems for piecewise – analytic functions and their applications in Besov spaces

ABSTRACT

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Boundary value problems in the theory of analytic functions constitute a research area of substantial scientific significance within both the general theory of functions and the theory of differential equations.

**Relevance of the research topic.** In the present dissertation, boundary value problems for continuous analytic (more precisely, piecewise analytic) functions are studied for the first time in the framework of Besov spaces. These spaces are embedded into the space of continuous (though not necessarily Hölder continuous) functions, which determines the scientific novelty of the research. Previously, such and similar problems were investigated under the assumption of Hölder continuity of the given functions.

The use of Besov spaces makes it possible to transfer properties of the density of differential operators, including singular integrals, directly to the operators themselves without loss of regularity. This, in turn, allows one to relax the conditions that guarantee the existence of solutions to classical differential equations. For example, such problems for the wave equation were studied in the works of O. V. Besov, while for elliptic systems on the plane they were considered in the works of N. K. Bliev.

Let a system of contours

$$L = L_0 + L_1 + \ldots + L_m.$$

be given. Suppose that the contour $L_0$ encloses all the remaining contours

$$L_1, L_2, \ldots, L_m.$$

Let $D^+$be the domain bounded by these contours, which is an $m+1$ connected domain lying inside the contour $L_0$ and outside the contours

$$L_1, L_2, \ldots, L_m.$$

Denote by $D^-$ the domain obtained as the complement of the set $D^+ \cup L$ in the whole complex plane. Thus, $D^-$ is the exterior complement of the domain $D^+$.

We assume that the origin of coordinates is located inside the domain $D^+$. The positive orientation of the contours is chosen so that the domain $D^+$ always lies to the left of the directed contour. Accordingly, the outer contour $L_0$ is oriented counterclockwise, whereas the inner contours

$$L_1, L_2, \dots, L_m$$

are oriented clockwise.

We consider the Besov space $B^r_{p,\theta}(L)$, where the parameters $r, p, \theta$ satisfy one of the following conditions:

$$\text{(I)} \quad 1 < p < 2, \theta = 1, r = 1/p,$$

$$\text{(II)} \quad 1 < p < 2, \theta \geq 1, r > 1/p,$$

$$\text{(III)} \quad p \geq 2, \theta \geq 1, r > 1 - 1/p.$$

In addition, we assume that the inequality

$$r + \frac{1}{p} - 1 < \nu \leq 1,$$

holds and that $L \in C^{1,\nu}$, where $C^{1,\nu}$ denotes the class of functions with a continuous derivative satisfying a Hölder condition of order $\nu$.
A Cauchy-type integral is defined by

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} d\tau.$$

The Sokhotski–Plemelj formulas take the form

$$\Phi^+(z) = \frac{1}{2}\varphi(t) + \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau, \quad t \in L,$$

$$\Phi^-(z) = -\frac{1}{2}\varphi(t) + \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - t} d\tau, \quad t \in L.$$

In case (I), the space $B^{1/p}_{p,1}(L)$ is embedded into the space of continuous functions $C(L)$, but is not embedded into the Hölder space

$$H_\beta(L), \ 0 < \beta < 1.$$

Thus, there exist functions $\varphi(t)$ belonging to the class $B^{1/p}_{p,1}(L)$ which are continuous on the contour but do not satisfy any Hölder condition. For such functions, the

Cauchy-type integral is continuous in the closed domain $\overline{D^+}$, and the Sokhotski–Plemelj formulas hold.

This result is important, since in the general case the Cauchy-type integral with an arbitrary continuous density is not necessarily continuous in the closed domain. Functions from the class $B_{p,1}^{1/p}(L)$ satisfy a Hölder-type inequality in the $L_p$ -norm.

Under conditions (II) and (III), we have the embedding

$$\varphi(t) \in B_{p,\theta}^{r}(L) \hookrightarrow H_{\beta}(L), \quad 0 < \beta < 1,$$

and in these cases the function $\Phi(z)$ defined by the Cauchy-type integral belongs to the space $B_{p,\theta}^{1+\alpha}(D^+)$. Accordingly, its generalized derivatives belong to the space $B_{p,\theta}^{\alpha}(D^+)$.

In this work, the properties of Cauchy-type integrals in multiply connected domains are studied within the framework of Besov spaces. The results obtained make it possible to solve singular integral equations in multiply connected domains and are applicable to the theory of complex boundary value problems in mathematical physics.

**Research objective.** The aim of this dissertation is to establish solvability conditions for boundary conjugation problems for piecewise analytic functions in multiply connected domains. These conditions are formulated in the class of continuous functions, and their validity in Besov spaces is proved. Such Besov spaces form a commutative algebra, which makes it possible to investigate their properties more deeply using the methods of functional analysis.

**Object of the research.** Boundary value (conjugation) problems for piecewise analytic functions in multiply connected domains in Besov spaces embedded into the class of continuous functions.

**Methods of research.** To solve boundary value problems for piecewise analytic functions in multiply connected domains within Besov spaces, the method of Cauchy-type integrals is employed.

**Theoretical and practical significance.** The theoretical significance of the research is associated with the development of new methods for the study of boundary value problems. Previously, such and related problems were studied under Hölder continuity assumptions. In the present work, these Hölder conditions are removed, and the possibility of seeking solutions in wider function spaces is explored. This allows one to reveal new properties of boundary value problems and to gain a deeper understanding of their solutions.

The practical significance of the research is related to mathematical modelling in various fields. Boundary value problems arise in many physical and engineering processes, including hydrodynamics, aerodynamics, electrical engineering, and the

theory of elasticity, where they play an important role. The use of Besov spaces enables one to obtain more accurate solutions and opens new possibilities for solving applied problems. Thus, the research not only expands the mathematical theory but also enhances its applicability in practice, which makes it particularly important both scientifically and practically.

**Scientific novelty. Main results submitted for defense.** This dissertation proposes a method for solving boundary value (conjugation) problems in Besov spaces in the class of continuous functions (without imposing Hölder continuity), which constitutes a key element of its novelty. It is well known that the solution of boundary conjugation problems is closely connected with the theory of singular integral equations involving Cauchy-type integrals. Of course, the theory of such singular integral equations can be developed without direct reference to boundary conjugation problems; however, the explicit consideration of these problems makes the theory significantly more transparent and illustrative.

For the first time, solutions to the homogeneous and inhomogeneous Riemann problems in a multiply connected domain have been obtained in Besov spaces, and the behaviour of these solutions at infinity has been investigated.

**Approbation.** The results of the dissertation have been presented and discussed at the following international conferences:

1. International congress on fundamental and applied sciences (ICFAS), Antalya, Turkey, 19-21 October.
2. VI International scientific conference “Non-local boundary value problems and related problems of mathematical biology, informatics and physics”, December 5-9, 2021, Nalchik, Kabardino-Balkarian Republic, Russia.
3. International scientific workshop “OTHA Fall 2021”, December 13-17, 2021, Southern Federal University. Rostov-on Don, Russia.
4. International conference “Numerical functional analysis-2021” of ICAAM, November 22-24, 2021, Istanbul, Turkey.
5. Workshop on Banach Spaces and Banach Lattices II, Instituto de Ciencias Mathematicas, May 9-13, 2022, Madrid, Spain.

**Reliability and validity of scientific statements, conclusions, and results.** The results obtained in this research are corroborated by proofs consistent with those established in earlier works by other authors, by publications in indexed international journals, and by materials of scientific conferences.

**Publications.**

1. N. K. Bliev and N. M. Yerkinbayev, “Riemann problem for multiply connected domain in Besov spaces,” Boundary Value Problems, vol. 2024(1), 70, p. 9, 2024. https://doi.org/10.1186/s13661-024-01883-x. (Q1)
2. N. K. Bliev and N. M. Yerkinbayev, “Boundary conjugation problem for piecewise analytic functions in Besov spaces,” Complex Variables and Elliptic

Equations, vol. 69(5), pp. 849–856, 2024. https://doi.org/10.1080/17476933.2023.2164887. (Q2)

3. N. K. Bliev, K. S. Tulenov, and N. M. Yerkinbayev, “Noetherian solvability and explicit solution of a singular integral equation with weighted carleman shift in besov spaces,” Complex Variables and Elliptic Equations, vol. 68, pp. 1264–1285, 2023. https://doi.org/10.1080/17476933.2022.2045975. (Q2)

**Structure of the dissertation.** The dissertation consists of an introduction, three chapters, a conclusion, and a list of references. The total length of the work is 79 pages. The bibliography comprises 35 references.